SHOWING PROPERTIES OF NUMERIC METHOD ON THE BASIS OF HOMOGRAPHIC FUNCTIONS

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Let function $f: R \to R$ be sufficiently smooth, $f(x_*) = 0$, $f'(x_*) \neq 0$, and let successive approximations $x_0, x_1, x_2, \dots$ be generated by a numeric method for solution of equation $f(x) = 0$. Recall well known asymptotic convergence behavior of certain three methods described in [1], namely Newton method, secant method and secant method with the recurrency $x_{k+1} = F(x_k, x_{k-1})$:

$$(x_{k+1} =)\ x_k - \frac{f(x_k)}{f'(x_k)} = x_* + (x_k - x_*)^2(\kappa + o(1)), \tag{1}$$

$$(x_{k+1} =)\ \frac{af(x_k) - x_k f(a)}{f(x_k) - f(a)} = x_* + (x_k - x_*)(q + o(1)), \tag{2}$$

$$(x_{k+1} =)\ \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})} = x_* + (x_{k-1} - x_*)(x_k - x_*)(\kappa + o(1)). \tag{3}$$

For homographic functions $\varphi_A(x) = (a_{11}x + a_{12})/(a_{21}x + a_{22})$, $\det(A) \neq 0$, the following is established. If $f = \varphi_A$ then formulae (1–3) hold exactly i.e. $o(1)$ can be thrown off; corresponding values for κ, q are $\kappa = f''(x_*)/2f'(x_*)$, $q = \kappa(a - x_*)$. These properties are specific for homographic functions: if we throw off $o(1)$ in any formula (1–3) and consider this formula as functional equation over unknown function f with independent variable x_k (formulae (1–2)) or with independent variables x_{k-1}, x_k (formula (3)), then we deduce that f is homographic.

It should be noted also that solving equation $x = \varphi(x)$ using simple iteration method i.e. creating sequence x_0, $x_1 = \varphi(x_0)$, $x_2 = \varphi(x_1) = \varphi(\varphi(x_0)), \dots$ is quite transparent in the case of homographic function $\varphi = \varphi_A$ because $\varphi_A(\varphi_A(x)) = \varphi_{A^2}(x)$, $\varphi_A(\varphi_A(\varphi_A(x))) = \varphi_{A^3}(x), \dots$ and powers of A are easy to calculate: $A^k = V\Lambda^k V^{-1}$ where $A = V\Lambda V^{-1}$ is canonical decomposition of 2 by 2 matrix A.

References

1. *Bakhvalov N.S.* Numerical methods. - Moscow: Nauka, 1975. 632 p.